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D-branes and KK-theory in type-I string theory

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Abstract: We analyse unstable D-brane systems in type-I string theory. Generalizing the proposal in hep-th/0108085, we give a physical interpretation for real KK-theory and claim that the D-branes embedded in a product space $X \times Y$ which are made from the unstable Dp-brane system wrapped on Y are classified by a real KK-theory group $\mathrm{KKO}^{p-1}(X, Y)$. The field contents of the unstable D-brane systems are systematically described by a hidden Clifford algebra structure.We also investigate the matrix theory based on non-BPS D-instantons and show that the spectrum of D-branes in the theory is exactly what we expect in type-I string theory, including stable non-BPS D-branes with $\mathbf{Z}_2$ charge. We explicitly construct the D-brane solutions in the framework of BSFT and analyse the physical property making use of the Clifford algebra.

Keywords: Superstrings and Heterotic Strings, D-branes, M(atrix) Theories.

Contents

1. Introduction

It has been argued that the D-brane charges are classified by K-theory [1, 2, 3]. Each element of K-theory is interpreted as a configuration of the gauge bundle and tachyon fields on a space-time filling unstable D-brane system, such as non-BPS D9-branes in type IIA [3] or D9-brane–anti D9-brane system in type IIB [2], describing a set of lower dimensional D-branes. This interpretation is obtained by generalizing the D-brane descent relations discussed in [4].

On the other hand, it is also known that we can construct higher dimensional D-branes from a lower dimensional D-brane system [5]–[10]. Therefore, it is natural to ask how we can classify the D-branes made by a lower dimensional unstable D-brane system. In the previous paper [11], we argued that K-homology groups are the groups which classify D-branes in matrix theories based on non-BPS D-instantons in type IIA or D-instanton–anti D-instanton system in type IIB, which we called K-matrix theory. The K-homology is a dual of K-theory and it turns out that this result is consistent with the classification of D-brane charge using K-theory. This means that the K-matrix theory correctly reproduces the D-brane spectrum expected from K-theory.

Furthermore, the argument is generalized to higher dimensional systems and we found that Kasparov's KK-theory groups are the relevant groups for the classification. To be more precise, suppose that the space-time manifold is a product space $X \times Y$, and consider stable D-branes made by the unstable D-brane system wrapped along Y. Then, the D-branes are naturally classified by the KK-theory group denoted by $KK^i(X,Y)$. (See section 2.1 for a brief review.) This group generalizes the above results. Actually, when the space X or Y is a point, the KK-theory group reduces to K-theory or K-homology, respectively. It is also shown in [11] that the spectrum of the D-branes does not depend on the choice of the unstable D-brane system, using some isomorphisms among the KK-theory groups.

In this paper, we generalize the argument to type-I string theory and consider type-I K-matrix theory, i.e. the matrix theory based on an infinite number of non-BPS D-instantons in type-I string theory, as well as the other unstable D-brane systems in type-I string theory. We are particularly interested in the construction of D-branes in these systems. As shown in [4, 2, 12], type-I string theory has non-trivial D-brane spectrum even when the space-time manifold is flat. In fact, the charge of flat Dp-branes transverse to $\mathbb{R}^{9-p}$ is classified by the real K-theory group[1] $\mathrm{KO}(\mathbb{R}^{9-p})$ [2], which is given in table 1.

p	-1	0	1	2	3	4	5	6	7	8	9
$\mathrm{KO}(\mathbb{R}^{9-p})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Table 1: The spectrum of flat Dp-branes in type-I theory, which is classified by the real K-theory group $\mathrm{KO}(\mathbb{R}^{9-p})$.

In particular, there are stable non-BPS D-branes with $\mathbb{Z}_2$ charges. So, it would be more challenging to explore this theory than the type-II sting theory. We will show that we correctly obtain this spectrum using type-I K-matrix theory.

It is easy to generalize the idea to higher dimensional systems. Actually, we will not restrict our arguments to the matrix theory, but consider D-branes made by unstable Dp-brane systems in type-I string theory, (i.e. non-BPS Dp-branes for $p = -1, 0, 2, 3, 4, 6, 7, 8$ and Dp-brane–anti Dp-brane system for $p = 1, 5, 9$). The gauge groups and the representation of tachyon fields on the world-volume of non-BPS Dp-branes are examined in [14], and the results are summarized in table 2.[2] It is also shown in [14, 2] that the K-theory group which classifies the charge of D-branes made by the descent relations from the unstable Dp-brane system is the real K-theory group $\mathrm{KO}^{p-1}(Y)$ ($\simeq \mathrm{KO}^{p-9}(Y)$), where Y is the world-volume manifold of the system.

[1]Here $\mathrm{KO}(\mathbb{R}^n)$ denotes the reduced K-theory group of S^n, $\widetilde{\mathrm{KO}}(S^n)$ [13].

[2]In this table, we omitted the massless scalar and tachyon modes from Dp-D9 strings for $p \geq 5$, since they do not affect the K-theory classification, as discussed in [14, 15]. So, we simply neglect them in this paper.

p	Gauge	Tachyon	Scalar	K-group
-1	U	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	adj.	$\mathrm{KO}^{-2}(pt)$
0	O	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$\Box\!\Box$	$\mathrm{KO}^{-1}(Y)$
1	$\mathrm{O}\times\mathrm{O}$	$(\Box,\Box)$	$(1,\Box\!\Box),(\Box\!\Box,1)$	$\mathrm{KO}^{0}(Y)$
2	O	$\Box\!\Box$	$\Box\!\Box$	$\mathrm{KO}^{1}(Y)$
3	U	$\Box\!\Box$	adj.	$\mathrm{KO}^{2}(Y)$
4	Sp	$\Box\!\Box$	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$\mathrm{KO}^{3}(Y)$
5	$\mathrm{Sp}\times\mathrm{Sp}$	$(\Box,\Box)$	$(1,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}),(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},1)$	$\mathrm{KO}^{4}(Y)$
6	Sp	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$\mathrm{KO}^{-3}(Y)$
7	U	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	adj.	$\mathrm{KO}^{-2}(Y)$
8	O	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$\Box\!\Box$	$\mathrm{KO}^{-1}(Y)$
9	$\mathrm{O}\times\mathrm{O}$	$(\Box,\Box)$	$--$	$\mathrm{KO}^{0}(Y)$

Table 2: The gauge groups, tachyons and scalar fields on the world-volume of type I unstable Dp-brane systems and corresponding K-theory groups.

From the argument analogous to that given in type-II case, it is quite natural to expect that the classification of type-I D-branes via the real K-theory is generalized by using real KK-theory, when we take into account the D-branes stretched along the directions transverse to the unstable Dp-brane system. As we will explain in section 2.2, the relevant KK-theory group which generalizes the K-theory group $\mathrm{KO}^{p-1}(Y)$ is the real KK-theory group $\mathrm{KKO}^{p-1}(X,Y)$, where X is the manifold transverse to the world-volume of the unstable Dp-brane system wrapped along Y. Note that when we forget about the transverse space X by setting X to be a point, the KK-theory group $\mathrm{KKO}^{p-1}(pt,Y)$ reduces to the real K-theory group $\mathrm{KO}^{p-1}(Y)$ and we correctly recover the K-theory results, though we describe the K-theory group using Fredholm operators as in [16, 17], which is slightly different from the description used in [14]. We will give a physical interpretation for each element of the KK-theory group in terms of the world-volume theory on the unstable Dp-brane system. As one can see from the table, the world-volume theory changes drastically as the dimension of the system changes. Therefore, it provides quite non-trivial check for our interpretation of the elements of KKO groups in terms of the world-volume field theory.

The paper is organized as follows. In section 2, we first review complex KK-theory and its physical interpretation in terms of unstable D-brane systems in type-II theory [11], and explain our proposal for type-I theory using the real KK-theory. The physical interpretation of the elements of real KK-theory is given in section 3. We argue that the field contents of the unstable D-brane system listed in table 2 nicely fit in with the definition of the real KK-theory groups. As we will see, the contents of table 2 can be easily reproduced by looking at real Clifford algebras, which are used in the definition of the KK-theory groups. Section 4 is mainly devoted to analyse type-I K-matrix theory. We will explicitly construct flat Dp-brane solutions in the framework of BSFT [18, 19, 20] for the type-I non-BPS D-instanton system. We examine the tension of the D-brane solutions and tachyon modes on them, making use of the Clifford algebra structure, and reproduce the expected property. Finally, we discuss further applications, such as the description of Chern-Simons terms using real superconnections, in section 5.

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2. KK-theory and D-branes

2.1 Complex KK-theory and type-II D-branes

First, we sketch the physical interpretation of complex KK-theory given in [11].

Let us consider type-II string theory on a product space $X \times Y$. As argued in [4, 2, 3], D-branes can be constructed as solitons on space-time filling unstable D-brane systems, such as non-BPS D9-branes in type IIA or D9-brane–anti D9-brane system in type IIB, wrapped on $X \times Y$. On the other hand, as mentioned in the introduction, we can construct higher dimensional D-branes from a lower dimensional unstable Dp-brane system. In particular, we can construct the space-time filling unstable D-brane system, as well as the D-brane solitons on it, from the lower dimensional unstable Dp-brane system. Therefore, any D-brane configurations can in principle be represented by the unstable Dp-brane system. So, let us suppose the dimension of Y is $p+1$ and consider unstable Dp-brane system wrapped on Y. Then, stable Dq-branes ($q \le p$) inside Y are contained as solitons, which represents a K-theory class of Y, in the system. Moreover, D-branes wrapped on a subspace of the transverse space X can also be constructed. For example, if we are interested in the configurations which are constant along Y, the construction of such D-branes is the same as that given in the K-matrix theory [11, 10]. Note that this construction is similar to those given in supersymmetric matrix theories [5, 6, 7], but here the tachyon fields play an essential role. These configurations are naturally classified by the analytic K-homology of X [11]. Therefore, in order to classify the D-brane configurations in the unstable Dp-brane system, we need a mathematical framework which generalizes both K-theory and K-homology.

The KK-theory is a generalization of both K-theory and K-homology, introduced by Kasparov [21, 22]. There are two kinds of complex KK-theory groups denoted by $KK^i(X,Y) = KK_i(C_0(X), C_0(Y))$ $(i = 0,1)$. Here X and Y are topological spaces (locally compact Hausdorff spaces), and $C_0(X)$ denotes the set of complex valued continuous functions on X vanishing at infinity.[3]

The KK-theory group $KK^1(X,Y)$ is defined[4] as the set of equivalence classes of triples $(\mathcal{H}, \phi, T)$, called odd Kasparov modules, where $\mathcal{H}$ $(= C_0(Y)^\infty)$ is a Hilbert space over $C_0(Y)$,[5] $\phi : C_0(X) \to \mathbf{B}(\mathcal{H})$ is a $*$-homomorphism and T is a self-adjoint operator in $\mathbf{B}(\mathcal{H})$ such that

$$T^2 - 1 \in \mathbf{K}(\mathcal{H}) \,, \qquad [T, \phi(a)] \in \mathbf{K}(\mathcal{H}) \qquad \text{for } {}^\forall a \in C_0(X) \,. \tag{2.1}$$

Here $\mathbf{B}(\mathcal{H})$ is the set of adjointable operators[6] on $\mathcal{H}$ and $\mathbf{K}(\mathcal{H})$ is the closure of 'finite rank'

[3]Note that every commutative C^*-algebra can be expressed as $C_0(X)$. One can define $KK_n(\mathcal{A}, \mathcal{B})$ for arbitrary (could be non-commutative) C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. The generalization to non-commutative cases is straightforward, though the physical interpretation is unclear in general.

[4]Here we assume that X and Y are compact, for simplicity.

[5]Hilbert space over a C^*-algebra $\mathcal{A}$, denoted by $\mathcal{A}^\infty$, is a Hilbert $\mathcal{A}$-module defined as $\mathcal{A}^\infty = \{(x_k) \in \prod_{n=1}^\infty \mathcal{A} \mid \sum_k x_k^* x_k \text{ converges in } \mathcal{A}\}$.

[6]An operator $T : \mathcal{H} \to \mathcal{H}$ is called adjointable if there is an operator $T^* : \mathcal{H} \to \mathcal{H}$ with $\langle Ta, b\rangle = \langle a, T^* b\rangle$ for all $a, b \in \mathcal{H}$. Here $\langle a, b\rangle = \sum_n a_n^* b_n$ for $a = (a_n), b = (b_n) \in \mathcal{H}$. Adjointable operators on a Hilbert $\mathcal{A}$-module are automatically $\mathcal{A}$ module homomorphism.

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operators in $\mathbf{B}(\mathcal{H})$.[7] Note that when Y is a point, $\mathcal{H}$ is a separable Hilbert space over $\mathbb{C}$, $\mathbf{B}(\mathcal{H})$ is the set of bounded linear operators on the Hilbert space $\mathcal{H}$, and $\mathbf{K}(\mathcal{H})$ is the set of compact operators on $\mathcal{H}$.

The equivalence relations are defined as follows. Two Kasparov modules $(\mathcal{H}_i, \phi_i, T_i)$ $(i = 0, 1)$ are called unitary equivalent when there is a unitary operator U in $\mathbf{B}(\mathcal{H}_0, \mathcal{H}_1)$ such that $T_0 = U^* T_1 U$ and $\phi_0(a) = U^* \phi_1(a) U$ for all $a \in C_0(X)$. They are called operator homotopic if $\mathcal{H}_0 = \mathcal{H}_1$, $\phi_0 = \phi_1$ and there is a norm continuous path between T_0 and T_1. We define a degenerate Kasparov module as the Kasparov module $(\mathcal{H}', \phi', T')$ satisfying $T'^2 - 1 = [T', \phi'(a)] = 0$ for all $a \in C_0(X)$. Then, in the definition of the KK-theory group, the two Kasparov modules $(\mathcal{H}_i, \phi_i, T_i)$ $(i = 0, 1)$ are defined to be equivalent if there are degenerate Kasparov modules $(\mathcal{H}'_i, \phi'_i, T'_i)$ $(i = 0, 1)$ such that $(\mathcal{H}_i \oplus \mathcal{H}'_i, \phi_i \oplus \phi'_i, T_i \oplus T'_i)$ $(i = 0, 1)$ are operator homotopic up to unitary equivalence.

This group classifies the solitonic configurations, which turn out to be D-branes embedded in the space-time $X \times Y$, in the system of an infinite number of non-BPS Dp-branes extended along Y and perpendicular to X. The space $\mathcal{H}$ is identified as a set of global sections of the infinite rank Chan-Paton bundle associated with the non-BPS Dp-branes. The unitary transformation acting on $\mathcal{H}$ is nothing but the gauge transformation of the system. The operator T is interpreted as the tachyon field on the non-BPS Dp-branes. The scalar fields Φ^i on the non-BPS Dp-branes, which represents their transverse positions, correspond to the operator $\phi(x^i)$, which are the image of the coordinate functions x^i under the $*$-homomorphism ϕ. Note that there are some delicate issues for the choice of the coordinate functions x^i, which we won't explain in detail here. (See [11].) All we need in the following is the fact that the scalar fields Φ^i are self-adjoint operators in the image of the $*$-homomorphism ϕ.

The condition (2.1) is related to the finiteness of the action. Here the tachyon is normalized such that the minimum of the potential is given by $T^2 = 1$. Hence, roughly speaking, the condition (2.1) represents that almost all the non-BPS Dp-branes are annihilated. (See [11] for more detail.)

The equivalence relations also have a nice physical interpretation. The unitary equivalence is nothing but the gauge equivalence, and the operator homotopy is just a continuous deformation of the tachyon configuration. The degenerate elements are interpreted as non-BPS D-instantons that would be annihilated by the tachyon condensation.

When X is a point, the KK-theory group $KK^1(pt, Y)$ is isomorphic to the K-theory group $K^1(Y)$. Therefore, we correctly reproduce the K-theory classification, if we are not interested in the the space X which is transverse to the non-BPS Dp-branes. In fact, the above interpretation reduces to that given by Witten in [17]. Similarly, when Y is a point, the KK-theory group $KK^1(X, pt)$ is isomorphic to the K-homology group $K_1(X)$, which is consistent with the classification of D-brane configurations in the K-matrix theory [11].

[7] A 'finite rank' operator is a linear span of the operators $\theta_{x,y}$ defined as $\theta_{x,y} z = x\langle y, z\rangle$ for $x, y, z \in \mathcal{H}$

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The other KK-theory group $KK^0(X,Y) = KK(X,Y)$ consists of the equivalence classes of $\mathbb{Z}_2$ graded triples $(\widehat{\mathcal{H}}, \widehat{\phi}, F)$, called even Kasparov modules, where

$$\widehat{\mathcal{H}} = \begin{pmatrix} \mathcal{H}^{(0)} \\ \mathcal{H}^{(1)} \end{pmatrix}, \qquad \widehat{\phi} = \begin{pmatrix} \phi^{(0)} & \\ & \phi^{(1)} \end{pmatrix}, \qquad F = \begin{pmatrix} & T^\dagger \\ T & \end{pmatrix}. \tag{2.2}$$

Here $\mathcal{H}^{(i)}$ $(i=0,1)$ are Hilbert spaces over $C_0(Y)$, $\phi^{(i)} : C_0(X) \to \mathbf{B}(\mathcal{H}^{(i)})$ $(i=0,1)$ are $*$-homomorphisms, and $T : \mathcal{H}^{(0)} \to \mathcal{H}^{(1)}$ is an adjointable operator satisfying

$$F^2 - 1 \in \mathbf{K}(\widehat{\mathcal{H}}), \quad [F, \widehat{\phi}(a)] \in \mathbf{K}(\widehat{\mathcal{H}}) \qquad \text{for } {}^\forall a \in C_0(X). \tag{2.3}$$

The grading is defined by the operator $\gamma = \mathrm{diag}(1,-1)$. Note that the operator $\widehat{\phi}(a)$ and F are chosen to be even and odd under this grading, respectively. The equivalence relation for $KK^0(X,Y)$ is defined in an analogous way as above. Note that the unitary operators, which are used in the definition of unitary equivalence, are required to be even with respect to the $\mathbb{Z}_2$ grading.

The physical interpretation of $KK^0(X,Y)$ is quite analogous to that given for $KK^1(X,Y)$. Here $KK^0(X,Y)$ classifies D-branes made by infinitely many Dp-brane–anti Dp-brane pairs wrapped on Y. $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ are interpreted as the infinite dimensional Chan-Paton Hilbert spaces of Dp-branes and anti Dp-branes, respectively, and T is the tachyon field created by the Dp–anti Dp strings.

It is worth noting that the odd Kasparov module $(\mathcal{H}, \phi, T)$ is also written as the even Kasparov module by setting $\mathcal{H}^{(0)} = \mathcal{H}^{(1)} = \mathcal{H}$, $\widehat{\phi} = \phi \otimes 1_2$ and $F = T \otimes \sigma_1$. In fact, these two KK-theory groups are related by the isomorphism[8]

$$KK_1(\mathcal{A}, \mathcal{B}) \simeq KK_0(\mathcal{A}, \mathcal{B} \otimes \mathbb{C}_1), \tag{2.4}$$

where $\mathbb{C}_1 = \mathbb{C} \oplus \mathbb{C} e_1$ is a (complex) Clifford algebra generated by an element e_1 satisfying $e_1^2 = 1$, which can be represented by associating e_1 to one of the Pauli matrices σ_1. A Clifford algebra has a natural ($\mathbb{Z}_2$-) grading which is given by regarding each generator as an odd element. The elements of $KK_0(C_0(X), C_0(Y) \otimes \mathbb{C}_1)$ is written as $(\mathcal{H} \otimes 1 \oplus \mathcal{H} \otimes e_1, \phi \otimes 1, T \otimes e_1)$, where $(\mathcal{H}, \phi, T)$ is an odd Kasparov module.

2.2 Real KK-theory and type-I D-branes

The real KK-theory is defined in the same way as above, except that we should use real objects, such as real C^*-algebras, real Hilbert spaces, real Clifford algebra, and so on. (See for example [23].) The analog of $KK(X,Y)$ $(= KK^0(X,Y))$ is denoted by $\mathrm{KKO}(X,Y) = \mathrm{KKO}(C_0(X, \mathbb{R}), C_0(Y, \mathbb{R}))$, where $C_0(X, \mathbb{R})$ is a set of continuous real functions on X vanishing at infinity. The KK-theory groups $\mathrm{KKO}^{-n}(X,Y)$, which we mainly consider, are defined by using the Clifford algebra. The Clifford algebra $\mathbb{C}^{p,q}$ is defined as an algebra over $\mathbb{R}$ generated by e_i $(i = 1, \ldots, p+q)$ satisfying

$$e_i e_j + e_j e_i = 0 \qquad (i \neq j) \tag{2.5}$$

$$e_i^2 = -1 \qquad (i = 1, \ldots, p) \tag{2.6}$$

$$e_i^2 = 1 \qquad (i = p+1, \ldots, p+q). \tag{2.7}$$

[8]Here we assume that the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are trivially graded. For general $\mathbb{Z}_2$-graded C^*-algebras, we should replace the tensor product $\otimes$ with the graded tensor product denoted by $\widehat{\otimes}$. (See [22, 23].)

(See appendix A for some more details about the Clifford algebra.) And we define

$$\mathrm{KKO}_{q-p+r-s}(\mathcal{A},\mathcal{B}) = \mathrm{KKO}(\mathcal{A}\otimes\mathbb{C}^{p,q},\mathcal{B}\otimes\mathbb{C}^{r,s})\,. \tag{2.8}$$

Note that one can show that the right hand side depends only on $q-p+r-s \pmod 8$, and the left hand side is well-defined. We also use the notation

$$\mathrm{KKO}^{-n}(X,Y) = \mathrm{KKO}_n(C_0(X,\mathbb{R}),C_0(Y,\mathbb{R})) \tag{2.9}$$

when the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are commutative and associated to topological spaces X and Y. In particular, when X is a point, they are related to the real K-theory as

$$\mathrm{KKO}^{-n}(pt,Y) = \mathrm{KO}^{-n}(Y)\,. \tag{2.10}$$

In type-I string theory, as we mentioned in the introduction, the K-theory group for the unstable Dp-brane system wrapped on Y, which classifies the charge of D-branes embedded in Y, is $\mathrm{KO}^{p-1}(Y)$ [14]. The analogous argument as that given for the type-II string theory in section 2.1 suggest that D-branes embedded in the space $X\times Y$ made from the unstable Dp-brane system wrapped on Y are classified by the KK-theory group $K\,\mathrm{KO}^{p-1}(X,Y)$. In the next section, we will give the physical interpretation of the elements of $K\,\mathrm{KO}^{p-1}(X,Y)$ in terms of the world-volume theory of the unstable Dp-brane system, and show more explicitly that this is the relevant group.

3. Real KK-theory and type-I non-BPS D-branes

Let us compare the real KK-theory groups and the world-volume theory of non-BPS Dp-branes. First, the elements of $\mathrm{KKO}^0(X,Y)$ can be interpreted in terms of Dp-brane–anti Dp-brane system ($p=1,9$) wrapped on Y in type-I string theory, using the exactly analogous argument as in the type-II case.

Then, let us consider $\mathrm{KKO}^{p-1}(X,Y)$ for $p\neq 1,9$. From (2.8), we have

$$\begin{aligned}\mathrm{KKO}^{-n}(X,Y) &= \mathrm{KKO}_n(C_0(X,\mathbb{R}),C_0(Y,\mathbb{R}))\,,\\ &= \begin{cases}\mathrm{KKO}(C_0(X,\mathbb{R}),C_0(Y,\mathbb{R})\otimes\mathbb{C}^{n,0}) & (n>0)\,,\\ \mathrm{KKO}(C_0(X,\mathbb{R}),C_0(Y,\mathbb{R})\otimes\mathbb{C}^{0,-n}) & (n<0)\,.\end{cases}\end{aligned} \tag{3.1}$$

We use (3.1) as the definition of the KK-groups $\mathrm{KKO}^{-n}(X,Y)$ and give a physical interpretation to them.

$\mathrm{KKO}(C_0(X,\mathbb{R}),C_0(Y,\mathbb{R})\otimes\mathbb{C}^{p,q})$ consists of equivalence classes of triples $(\widehat{\mathcal{H}},\widehat{\phi},F)$, where $\widehat{\mathcal{H}}$ is the real Hilbert space over $C_0(Y,\mathbb{R})\otimes\mathbb{C}^{p,q}$, $\widehat{\phi}: C_0(X,\mathbb{R})\to\mathbf{B}(\widehat{\mathcal{H}})$ is a $*$-homomorphism, and F is a self-adjoint operator in $\mathbf{B}(\widehat{\mathcal{H}})$. We also require that $\widehat{\phi}(a)$ ($a\in C_0(X,\mathbb{R})$) is even and F is odd with respect to the $\mathbb{Z}_2$-grading. The equivalence relations are again given by unitary equivalence, operator homotopy and addition of degenerate elements, which are defined in analogous way as explained in section 2.1 for the complex KK-theory. Here the unitary equivalence is given by even unitary operators in $\mathbf{B}(\widehat{\mathcal{H}})$.

Since $\widehat{\mathcal{H}} \simeq \mathcal{H} \otimes \mathbb{C}^{p,q}$, where $\mathcal{H} = C_0(Y,\mathbb{R})^\infty$, one can show that $\mathbf{B}(\widehat{\mathcal{H}}) \simeq \mathbf{B}(\mathcal{H}) \otimes \mathbb{C}^{p,q}$. Therefore, the operator F and $\widehat{\phi}(a)$ for $a \in C_0(X,\mathbb{R})$ can be expressed as

$$F = \sum_{v_n \in \mathbb{C}^{p,q}_{\text{odd}}} T_n v_n\,, \tag{3.2}$$

$$\widehat{\phi}(a) = \sum_{w_n \in \mathbb{C}^{p,q}_{\text{even}}} \Phi_n w_n\,, \tag{3.3}$$

where $T_n, \Phi_n \in \mathbf{B}(\mathcal{H})$. Here $\mathbb{C}^{p,q}_{\text{even}}$ and $\mathbb{C}^{p,q}_{\text{odd}}$ denote the sets of even and odd elements in $\mathbb{C}^{p,q}$ spanned by the basis w_n and v_n, respectively.

We will show in the following that the operators $\widehat{\phi}(a)$ ($a \in C_0(X,\mathbb{R})$) and F correctly behave as the scalar and tachyon fields listed in table 2, respectively. For this purpose, we can restrict our consideration to the configurations which are constant along Y without any loss of generality. So, we will set Y to be a point for simplicity. In this case, $\mathcal{H}$ is just a separable Hilbert space over $\mathbb{R}$ and $\mathbf{B}(\mathcal{H})$ is the set of bounded linear operators on the Hilbert space $\mathcal{H}$. Thanks to the Bott periodicity ($\mathrm{KKO}^n(X,Y) \simeq \mathrm{KKO}^{n\pm 8}(X,Y)$), it is sufficient to consider $\mathrm{KKO}^n(X,Y)$ with $-3 \le n \le 4$. Therefore, we will consider $\mathrm{KKO}^{-n}(X,pt) = \mathrm{KKO}(C_0(X,\mathbb{R}),\mathbb{C}^{n,0})$ ($n=1,2,3$) and $\mathrm{KKO}^n(X,pt) = \mathrm{KKO}(C_0(X,\mathbb{R}),\mathbb{C}^{0,n})$ ($n=1,2,3,4$) in the following.

3.1 Non-BPS D0, D8-brane: $\mathrm{KKO}^{-1}(X,Y)$

Let us consider $\mathrm{KKO}^{-1}(X,pt) = \mathrm{KKO}(C_0(X,\mathbb{R}),\mathbb{C}^{1,0})$. The generator e_1 of $\mathbb{C}^{1,0}$ satisfies $e_1^2 = -1$, and it can be represented by i in $\mathbb{C}$. Thus $\mathbb{C}^{1,0} = \mathbb{R} \oplus \mathbb{R}i = \mathbb{C}$, where real (imaginary) part consists of the even (odd) elements.

The gauge transformation is identified as the unitary transformation on $\widehat{\mathcal{H}}$ which is even with respect to the $\mathbb{Z}_2$-grading. In this case, they are real unitary operators, which implies that the gauge group is the orthogonal group $\mathrm{O}(\infty)$.

The tachyon is identified as an odd operator $F = iT$, where T is a real operator, with the self-adjoint condition $F^\dagger = F$. Therefore, T should be an anti-symmetric operator ($T^T = -T$) which behaves as the anti-symmetric tensor representation $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ under the gauge transformation.

The scalar fields are even elements which means that they are real operators. Furthermore, they should be self-adjoint since they are in the image of the $*$-homomorphism $\widehat{\phi}$. Then the scalar fields behave as the symmetric tensor representation $\Box\Box$ of the gauge group.

These results are consistent with the world-volume theory of non-BPS D0 or D8-branes.

3.2 Non-BPS D(−1), D7-brane: $\mathrm{KKO}^{-2}(X,Y)$

Let us consider $\mathrm{KKO}^{-2}(X,pt) = \mathrm{KKO}(C_0(X,\mathbb{R}),\mathbb{C}^{2,0})$. The generators of $\mathbb{C}^{2,0}$ are represented by the elements of the quaternion as $e_1 = i$, $e_2 = j$, where i and j are two of the generators of the quaternion i,j,k which are anti-hermitean and satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. Hence, $\mathbb{C}^{2,0} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k = \mathbf{H}$, in which even elements are $\mathbb{C}^{2,0}_{\text{even}} = \mathbb{R} \oplus \mathbb{R}k$ and odd elements are $\mathbb{C}^{2,0}_{\text{odd}} = \mathbb{R}i \oplus \mathbb{R}j$.

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Since the unitary transformation is given by an even element which is of the form $g = g_1 + g_2 k$, where g_1 and g_2 are real operators, satisfying $g^\dagger g = 1$, the gauge group consists of (complex) unitary operators $\mathrm{U}(\infty)$.

The tachyon is an odd element $F = T_1 i - T_2 j = i(T_1 + T_2 k)$ satisfying $F^\dagger = F$, where T_1 and T_2 are real operators. Let us define $T \equiv T_1 + T_2 k$, then T is a complex anti-symmetric operator satisfying $T^T = -T$. The transformation of tachyon under the gauge group is given by

$$F = iT \to g^\dagger F g = i g^T T g\,, \tag{3.4}$$

from which we can see that T is transformed as the anti-symmetric tensor representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, as expected.

The scalar fields are even self-adjoint operators. This means that the scalar fields are complex Hermite operators, and they belong to the adjoint representation of the gauge group $\mathrm{U}(\infty)$.

These results are consistent with the world-volume theory of non-BPS D(-1) or D7-branes.

3.3 Non-BPS D6-brane: $\mathrm{KKO}^{-3}(X, Y)$

Let us consider $\mathrm{KKO}^{-3}(X, pt) = \mathrm{KKO}(C_0(X, \mathbb{R}), \mathbb{C}^{3,0})$. The generators of $\mathbb{C}^{3,0}$ are faithfully represented as

$$e_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, e_2 = \begin{pmatrix} j & \\ & -j \end{pmatrix}, e_3 = \begin{pmatrix} k & \\ & -k \end{pmatrix}, \tag{3.5}$$

where i, j, k are the generators of quaternion. Even elements of $\mathbb{C}^{3,0}$ are of the form

$$\begin{pmatrix} a + bi + cj + dk & \\ & a + bi + cj + dk \end{pmatrix} = \begin{pmatrix} a' & \\ & a' \end{pmatrix}, \tag{3.6}$$

and odd elements are

$$\begin{pmatrix} a + bi + cj + dk & \\ & -a - bi - cj - dk \end{pmatrix} = \begin{pmatrix} a' & \\ & -a' \end{pmatrix}, \tag{3.7}$$

where $a, b, c, d \in \mathbb{R}$ and $a' = a + bi + cj + dk \in \mathbf{H}$ is a quaternion.

The gauge transformation is given by an even element

$$g = \begin{pmatrix} g' & \\ & g' \end{pmatrix}, \tag{3.8}$$

with $g^\dagger g = 1$, where $g' = g_a + g_b i + g_c j + g_d k$ is a quaternionic operator. It is easy to see that the quaternionic unitary operator g' is an element of Sp group. (See appendix B.) Thus the gauge group is $\mathrm{Sp}(\infty)$ as expected.

The tachyon operator is a self-adjoint odd element of the Clifford algebra, and hence we can set

$$F = \begin{pmatrix} T & \\ & -T \end{pmatrix}, \qquad T^\dagger = T\,, \tag{3.9}$$

where $T = T_0 + T_1 i + T_2 j + T_3 k$ is a quaternionic operator. As shown in appendix B, this behaves as the anti-symmetric tensor representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of the gauge group.

The scalar fields are even self-adjoint elements,

$$\Phi = \begin{pmatrix} \Phi' & \\ & \Phi' \end{pmatrix}, \qquad \Phi'^{\dagger} = \Phi', \tag{3.10}$$

where Φ' is a quaternionic operator. Thus they also transform as the anti-symmetric tensor representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ under the gauge transformation.

All these results are consistent with the world-volume theory of non-BPS D6-branes.

3.4 D5–anti D5 system: $\mathrm{KKO}^{-4}(X, Y)$

Let us consider $\mathrm{KKO}^{-4}(X, pt) = \mathrm{KKO}(C_0(X, \mathbb{R}), \mathbb{C}^{4,0})$. The generators for the Clifford algebra $\mathbb{C}^{4,0}$ are represented as

$$e_1 = \begin{pmatrix} & i \\ i & \end{pmatrix}, \qquad e_2 = \begin{pmatrix} & j \\ j & \end{pmatrix}, \qquad e_3 = \begin{pmatrix} & k \\ k & \end{pmatrix}, \qquad e_4 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \tag{3.11}$$

where i, j, k are the generators of quaternion. Then, the even elements are of the form

$$\begin{pmatrix} a + bi + cj + dk & \\ & e + fi + gj + hk \end{pmatrix} \tag{3.12}$$

and odd elements are

$$\begin{pmatrix} & a + bi + cj + dk \\ e + fi + gj + hk & \end{pmatrix} \tag{3.13}$$

where $a, b, c, d, e, f, g, h \in \mathbb{R}$.

The gauge group consists of even unitary operators, which is of the form

$$g = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, \tag{3.14}$$

where g_1 and g_2 are quaternionic operators satisfying $g_1^{\dagger} g_1 = g_2^{\dagger} g_2 = 1$. As explained in appendix B, the unitary quaternionic operators g_1 and g_2 are elements of Sp group. Therefore, the gauge group is $\mathrm{Sp}(\infty) \times \mathrm{Sp}(\infty)$,

The tachyon operator is a self-adjoint odd operator, which is of the form

$$F = \begin{pmatrix} & T^{\dagger} \\ T & \end{pmatrix}, \tag{3.15}$$

where T is a quaternionic operator. It transforms as

$$F \to g^{\dagger} F g = \begin{pmatrix} & g_1^{\dagger} T^{\dagger} g_2 \\ g_2^{\dagger} T g_1 & \end{pmatrix}, \tag{3.16}$$

under the gauge transformation, which show that the operator T transform as the bi-fundamental representation of the gauge group $\mathrm{Sp}(\infty) \times \mathrm{Sp}(\infty)$.

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The scalar fields Φ are even operators of the form

$$\Phi = \begin{pmatrix} \Phi_1 & \\ & \Phi_2 \end{pmatrix}, \tag{3.17}$$

where Φ_1 and Φ_2 are self-adjoint quaternionic operators, which means that Φ_i $(i = 1, 2)$ belong to anti-symmetric tensor representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of the gauge group. (See appendix B.)

These results are consistent with the world-volume theory of D5-brane–anti D5-brane system.

3.5 Non-BPS D2-brane: $\mathrm{KKO}^1(X, Y)$

Let us consider $\mathrm{KKO}^1(X, pt) = \mathrm{KKO}(C_0(X, \mathbb{R}), \mathbb{C}^{0,1})$. The generator e_1 of $\mathbb{C}^{0,1}$ is represented by $e_1 = \sigma_3$. The even element is of the form

$$\begin{pmatrix} a & \\ & a \end{pmatrix}, \tag{3.18}$$

and the odd element is

$$\begin{pmatrix} a & \\ & -a \end{pmatrix} \tag{3.19}$$

for $a \in \mathbb{R}$.

The gauge group consists of even elements, and hence real unitary operators, namely $\mathrm{O}(\infty)$.

The tachyon is an odd element which is of the form $F = T\sigma_3$, where T is a real operator, with $F^\dagger = F$, which means that T is the symmetric tensor representation $\square\square$ of the gauge group.

The scalar fields are even self-adjoint elements, which also belong to the symmetric tensor representation $\square\square$ of the gauge group $\mathrm{O}(\infty)$.

These results are consistent with the world-volume theory of non-BPS D2-branes.

3.6 Non-BPS D3-brane: $\mathrm{KKO}^2(X, Y)$

Let us consider $\mathrm{KKO}^2(X, pt) = \mathrm{KKO}(C_0(X, \mathbb{R}), \mathbb{C}^{0,2})$. The generators of $\mathbb{C}^{0,2}$ are represented by $e_1 = \sigma_1$ and $e_2 = \sigma_2$. An even element is of the form

$$\begin{pmatrix} a + ib & \\ & a - ib \end{pmatrix}, \tag{3.20}$$

and an odd element is

$$\begin{pmatrix} & a - ib \\ a + ib & \end{pmatrix}, \tag{3.21}$$

where $a, b \in \mathbb{R}$.

The gauge group consists of even elements of the form

$$g = \begin{pmatrix} g_a + ig_b & \\ & g_a - ig_b \end{pmatrix} \equiv \begin{pmatrix} g' & \\ & \overline{g'} \end{pmatrix}. \tag{3.22}$$

Therefore the gauge group is $\mathrm{U}(\infty)$.

The tachyon operator is an odd element

$$F = \begin{pmatrix} & \overline{T} \\ T & \end{pmatrix}, \tag{3.23}$$

where $T = T_a + T_b i$ is a complex operator. The self-adjoint condition $F^\dagger = F$ implies $T = T^T$, which means that the tachyon belongs to the symmetric tensor representation $\square\square$ of the gauge group. In fact, the gauge transformation $F \to g^\dagger F g$ becomes

$$T \to {g'}^T T g' , \tag{3.24}$$

which is the correct transformation of the symmetric tensor representation $\square\square$.

The scalars are even self-adjoint elements, and hence they belong to the adjoint representation of the gauge group $\mathrm{U}(\infty)$.

These results are consistent with the world-volume theory of non-BPS D3-branes.

3.7 Non-BPS D4-brane: $\mathrm{KKO}^3(X, Y)$

Let us consider $\mathrm{KKO}^3(X, pt) = \mathrm{KKO}(C_0(X, \mathbb{R}), \mathbb{C}^{0,3})$. The generators of $\mathbb{C}^{0,3}$ are represented as $e_1 = \sigma_1$, $e_2 = \sigma_2$ and $e_3 = \sigma_3$. The even elements are

$$a + bi\sigma_1 + ci\sigma_2 + di\sigma_3 , \tag{3.25}$$

and odd elements are

$$ai + b\sigma_1 + c\sigma_2 + d\sigma_3 , \tag{3.26}$$

where $a, b, c, d \in \mathbb{R}$.

The gauge transformation is given by even unitary operator, which is of the form

$$\begin{aligned} g &= g_a + g_b i\sigma_1 + g_c i\sigma_2 + g_d i\sigma_3 \\ &= \begin{pmatrix} g_a + ig_d & g_c + ig_b \\ -g_c + ig_b & g_a - ig_d \end{pmatrix}, \end{aligned} \tag{3.27}$$

satisfying $g^\dagger g = 1$. This is nothing but the Sp group. (See appendix B.)

It may be convenient to represent $i\sigma_i$, by the generators of quaternion as $i\sigma_1 \to i$, $i\sigma_2 \to j$ and $i\sigma_3 \to -k$. Then the gauge group is represented as quaternionic unitary operator, which we encountered in (3.8).

Then, the tachyon operator is an odd element $F = iT_0 + T_1\sigma + T_2\sigma_2 + T_3\sigma_3 \equiv iT$ satisfying $F^\dagger = F$. If we express the operator T in the quaternionic representation, we see that

$$T = T_0 - T_1 i - T_2 j + T_3 k , \qquad T^\dagger = -T , \tag{3.28}$$

which transforms as

$$T \to g^\dagger T g , \tag{3.29}$$

under the gauge transformation. Therefore, as shown in appendix B, the tachyon is transformed as the adjoint representation ($\square\square$) of the gauge group $\mathrm{Sp}(\infty)$.

The scalars are even elements. Using the quaternionic representation, it can be written as

$$\Phi = \Phi_a + \Phi_b i + \Phi_c j + \Phi_d k \,, \qquad \Phi^\dagger = \Phi \,, \tag{3.30}$$

which transform as the anti-symmetric tensor representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of the gauge group, as shown in appendix B.

These results are consistent with the world-volume theory of non-BPS D4-branes.

4. An explicit construction of flat Dp-branes

Let us make things more explicit in some simple situations. In this section, we consider flat D-branes in flat space-time, mainly using the matrix theory based on an infinite number of non-BPS D-instantons in type-I string theory, which we call type-I K-matrix theory.

4.1 Flat Dp-branes in type-I K-matrix theory

The type-I K-matrix theory resembles IIB matrix theory [7] in many respects. In fact, it is a matrix theory with unitary gauge symmetry, and the field contents include scalar fields and fermions which constitute vector and spinor representation of the Lorentz group $\mathrm{SO}(1,9)$, respectively, and transform as the adjoint representation of the gauge group. But there are some important differences. First, we have some extra matters including a tachyon, and then, we have to take the size of the matrices to be infinity from the beginning in order that we can create an arbitrary number of non-BPS D-instantons.

As we have argued, D-branes in the type-I K-matrix theory are classified by $\mathrm{KKO}^{-2}(X, pt)$. Let us consider flat Dp-branes extended in the $x^0, \ldots, x^p$ directions in flat space-time $\mathbb{R}^{10}$ in the type-I K-matrix theory. These configurations are classified by $\mathrm{KKO}^{-2}(\mathbb{R}^{p+1}, pt)$.[9]

Let us first consider the $p = -1$ case. The group $\mathrm{KKO}^{-2}(pt, pt)$ consists of homotopy classes of the anti-symmetric operator T acting on a Hilbert space $\mathcal{H}$ (over $\mathbb{C}$), satisfying $T^\dagger T - 1 \in \mathbf{K}(\mathcal{H})$. This condition implies that $\ker T$ is a finite dimensional vector space. Since T is anti-symmetric, $\dim \ker T \pmod 2$ is invariant under small perturbation of the operator T. In fact, we have $\mathrm{KKO}^{-2}(pt, pt) = \mathrm{KO}^{-2}(pt) = \mathbb{Z}_2$. Therefore, a single non-BPS D-instanton is stable, while a pair of non-BPS D-instantons can be annihilated, in agreement with the results in [2].

For generic p, we obtain

$$\mathrm{KKO}^{-2}(\mathbb{R}^{p+1}, pt) = \mathrm{KO}(\mathbb{R}^{9-p}) = \begin{cases} \mathbb{Z} & (p = 1, 5, 9 \pmod 8) \\ \mathbb{Z}_2 & (p = -1, 0, 7, 8 \pmod 8) \\ 0 & (\text{others}) \end{cases} \tag{4.1}$$

[9] Here $\mathbb{R}^{p+1}$ is not compact, but $\mathrm{KKO}^{-2}(\mathbb{R}^{p+1}, pt) = \mathrm{KKO}_2(C_0(\mathbb{R}^{p+1}, \mathbb{R}), \mathbb{R})$ is well-defined. It satisfies $\mathrm{KKO}^{-2}(S^{p+1}, pt) = \mathrm{KKO}^{-2}(\mathbb{R}^{p+1}, pt) \oplus \mathrm{KKO}^{-2}(pt, pt)$, where S^{p+1} is the one point compactification of $\mathbb{R}^{p+1}$. The left hand side $\mathrm{KKO}^{-2}(S^{p+1}, pt)$ classifies D-branes on S^{p+1}, and $\mathrm{KKO}^{-2}(pt, pt)$ in the right hand side classifies D-instantons sitting at a point in the space-time. The only possible stable D-branes on S^{p+1} are Dp-branes wrapped on the S^{p+1} and the D-instantons. Thus, we interpret $\mathrm{KKO}^{-2}(\mathbb{R}^{p+1}, pt)$ as the group which classifies the Dp-branes.

using the isomorphism

$$\mathrm{KKO}^k(X,Y) = \mathrm{KKO}^{k-n}(X\times\mathbb{R}^n, Y) = \mathrm{KKO}^{k+m}(X, Y\times\mathbb{R}^m)\,. \tag{4.2}$$

Hence we have $\mathbb{Z}_2$ charge Dp-branes ($p=-1,0,7,8$) and $\mathbb{Z}$ charge Dp-branes ($p=1,5,9$), which is exactly what we expect from the K-theory analysis [2], i.e. table 1.

The explicit configuration representing Dp-brane can be obtained by finding the configuration in type-IIB K-matrix theory (i.e. the matrix theory based on D-instanton–anti D-instanton system in type IIB theory) which survive after projecting out the unwanted components of the matrices in type-I theory.

Recall that there are ten pairs of scalars Φ^μ, $\overline{\Phi}^\mu$ ($\mu=0,1,\ldots,9$), which represent the position of D-instantons and anti D-instanton respectively, together with a tachyon T in the IIB K-matrix theory. A configuration representing a Dp-brane in the IIB K-matrix theory [10, 11] is given by

$$T = u\sum_{\alpha=0}^{p}\widehat{p}_\alpha\otimes\gamma^\alpha, \tag{4.3}$$

$$\Phi^\alpha = \overline{\Phi}^\alpha = \widehat{x}^\alpha\otimes 1 \qquad (\alpha=0,\cdots,p)\,, \tag{4.4}$$

$$\Phi^i = \overline{\Phi}^i = 0 \qquad (i=p+1,\cdots,9)\,, \tag{4.5}$$

which act on the Hilbert space $L^2(\mathbb{R}^{p+1})\otimes S$, where S is the vector space on which the matrices γ^α are represented. Here $\widehat{x}^\alpha$ is defined by multiplication of x^α and $\widehat{p}_\alpha = -i\partial_\alpha = -i\partial/\partial x^\alpha$ is a differential operator, both acting on $L^2(\mathbb{R}^{p+1})$, and

$$\Gamma^\alpha \equiv \begin{pmatrix} & \gamma^{\alpha\dagger} \\ \gamma^\alpha & \end{pmatrix} \tag{4.6}$$

are SO$(p+1)$ gamma matrices satisfying $\{\Gamma^\alpha,\Gamma^\beta\} = 2\delta^{\alpha\beta}$. (4.3) can also be written as

$$F \equiv \begin{pmatrix} & T^\dagger \\ T & \end{pmatrix} = u\sum_{\alpha=0}^{p}\widehat{p}_\alpha\otimes\Gamma^\alpha = u\sum_{\alpha=0}^{p}\partial_\alpha\otimes\widehat{\Gamma}^\alpha, \tag{4.7}$$

where we have defined $\widehat{\Gamma}^\alpha = -i\Gamma^\alpha$. Note that if p is even, we can choose γ^α to be hermitean, which form the irreducible SO$(p+1)$ gamma matrices. The size of the matrices γ^α are listed in table 3. It can be shown that this configuration becomes an exact Dp-brane solution of the BSFT action for type-IIB D-instanton–anti D-instanton system [18, 19, 20], if we take $u\to\infty$ [10].[10] The solution represents a BPS Dp-brane for odd p and a non-BPS Dp-brane for even p. The tension and RR-charge for the solution can be calculated in the BSFT framework and it is also shown in [10] that they exactly reproduce the expected value.

p	0	1	2	3	4	5	6	7	8	9
size	1	1	2	2	4	4	8	8	16	16

Table 3: The size of the matrices γ^α in (4.3) used for the tachyon configuration representing type-IIB Dp-branes.

The part of the action of the type-I non-BPS D-instantons which include only Φ^μ and T is obtained by projecting out the unwanted components of the matrices Φ^μ $\overline{\Phi}^\mu$ and T

[10]Note that the minimum of the tachyon potential of the BSFT action is given by $TT^\dagger = T^\dagger T = \infty\cdot 1$. This is the reason that we chose an unbounded operator for the tachyon in (4.3). If one wish to normalize the tachyon so that the minimum is given by $T'T'^\dagger = T'^\dagger T' = 1$, as in section 2.1, one can redefine the tachyon as $T' = T/\sqrt{1+T^\dagger T}$.

from the type-IIB BSFT action, at least at the tree level. Therefore, the solution of type-IIB K-matrix theory satisfying $\Phi^\mu = \overline{\Phi}^\mu$ and $T^T = -T$ is automatically a solution of type-I K-matrix theory. In order for T to be anti-symmetric, γ^α should be symmetric matrices, since the differential operators $\widehat{p}_\alpha$ are anti-symmetric. It is easy to find a representation of $\mathrm{SO}(p+1)$ gamma matrices Γ^α of the form (4.6) with symmetric γ^α. First, note that one can always set Γ^0 as

$$\Gamma^0 = 1 \otimes \sigma^2 \,, \tag{4.8}$$

using unitary transformation. Then, the condition $\{\Gamma^\alpha, \Gamma^\beta\} = 2\delta^{\alpha\beta}$ implies

$$\Gamma^i = \gamma_p^i \otimes \sigma^1 \qquad (i = 1, \ldots, p) \,, \tag{4.9}$$

where γ_p^i are $\mathrm{SO}(p)$ gamma matrices represented as real symmetric matrices. Note also that we can identify $-i\sigma^2$ and $-i\sigma^1$ as the generators e_1 and e_2 of the Clifford algebra $\mathbb{C}^{2,0}$, and the operator F in (4.7) can be written of the form

$$F = \begin{pmatrix} & T^\dagger \\ T & \end{pmatrix} = u\left(\partial_0\, e_1 + \partial_i \gamma_p^i\, e_2\right) . \tag{4.10}$$

The minimum size of the matrices γ^α are listed in table 4, which can be obtained from table 6 in appendix A, since the $\mathrm{SO}(p)$ gamma matrices γ_p^i form a real representation of $\mathbb{C}^{0,p}$. Comparing table 3 and table 4, we see that the size of the tachyon in type I with $p = 3, 4, 5, 6, 7$ is twice the size in type IIB. This implies that the tension of a Dp-brane in type-I theory are twice that in type-IIB theory for $p = 3, 4, 5, 6, 7$, while they are the same for $p = 0, 1, 2, 8, 9$, up to the common factor $1/\sqrt{2}$. This result is consistent with the construction of D-branes in type-I theory as given in [2, 14]. Namely, a Dp-brane in type-I theory is given by the unoriented projection of a single Dp-brane ($p = 0, 1, 2, 8, 9$), two Dp-branes ($p = 4, 5, 6$) or a Dp-brane–anti Dp-brane pair ($p = -1, 3, 7$) in type-IIB theory, as one can read from the gauge group listed in table 2.

p	0	1	2	3,	4	5	6	7	8	9
size	1	1	2	4	8	8	16	16	16	16

Table 4: The size of the matrices γ^α for the tachyon configuration representing type-I Dp-branes.

Let us next consider the tachyonic mode around the Dp-brane solution (4.10). Suppose that the tachyon F is the sum $F_0 + F'$ of the solution F_0 and the fluctuation F' satisfying

$$\{F_0, F'\} = 0 \,, \qquad {F'}^\dagger = F' \,. \tag{4.11}$$

Then, $F^2 = F_0^2 + F'^2$ implies that F' has negative mass squared, which means that F' is the tachyonic mode on the Dp-brane. In fact, inserting (4.11) into the BSFT action for the non-BPS D-instanton system, one can easily see that the potential for the fluctuation F' is exactly what we expect for the Dp-brane. Such fluctuation around the solution (4.10) is of the form

$$F' = T' \widehat{\gamma}\, e_2 \,, \tag{4.12}$$

where T' is a real parameter and $\widehat{\gamma}$ is a real matrix, whose size is the same as that of γ_p^i listed in table 4, satisfying

$$\{\widehat{\gamma}, \gamma_p^i\} = 0 \,, \qquad \widehat{\gamma}^\dagger = -\widehat{\gamma} \,, \qquad \widehat{\gamma}^2 = -1 \,. \tag{4.13}$$

Note that γ_p^i $(i = 1, \ldots, p)$ together with $\widehat{\gamma}$ make up a real representation of $\mathbb{C}^{1,p}$. In order that such $\widehat{\gamma}$ exists, there must be a real representation of $\mathbb{C}^{1,p}$ whose dimension is the same as the size of γ_p^i. The dimension of the irreducible real representation of $\mathbb{C}^{1,p}$ is listed in table 5, which is obtained using the identity $\mathbb{C}^{1,p} = M_2(\mathbb{C}^{0,p-1})$ (See (A.4).) and table 6. Therefore, comparing table 4 and table 5, we conclude that there is a tachyonic mode for $p = 2, 3, 4, 6$, which agrees with the standard result first derived in [2].

p	0	1	2	3	4	5	6	7	8	9
dim.	2	2	2	4	8	16	16	32	32	32

Table 5: The dimension of the irreducible real representation of $\mathbb{C}^{1,p}$.

Then, what happens if there are two Dp-branes for $p = 0, 1, 5, 7, 8, 9$? The solution representing two Dp-branes is obtained by simply tensoring the rank two unit matrix 1_2 with the solution (4.10) as

$$F = u\, 1_2 \otimes \left(\partial_0\, e_1 + \partial_i \gamma_p^i\, e_2\right). \tag{4.14}$$

In this case, the fluctuation

$$F' = T'\, \epsilon \otimes \gamma^{p+1} e_2 \tag{4.15}$$

satisfies the condition (4.11). Here $\epsilon = i\sigma_2$ and γ^{p+1} is a real matrix, whose size is the same as that of γ_p^i, satisfying

$$\{\gamma^{p+1}, \gamma_p^i\} = 0\,, \qquad \gamma^{p+1\dagger} = \gamma^{p+1}\,, \qquad (\gamma^{p+1})^2 = 1\,. \tag{4.16}$$

Namely, γ_p^i $(i = 1, \ldots, p)$ together with γ^{p+1} make up a real representation of $\mathbb{C}^{0,p+1}$. The analogous argument as above implies that there is a tachyonic mode if the dimension of the real representation of $\mathbb{C}^{0,p+1}$, which is given by shifting p to $p+1$ in table 4, is equal to the size of the matrices γ_p^i listed in table 4. Hence, we conclude that two coincident non-BPS Dp-branes with $p = 0, 7, 8$ have tachyon modes, which again reproduces the results in [2]. All these results are consistent with the spectrum of D-branes in type-I theory (table 1).

4.2 Flat Dp-branes in the unstable Dq-brane system

Now we generalize the above consideration and describe the construction of flat Dp-branes from unstable Dq-brane system with $q \geq 0$ in type-I theory. Let us consider the unstable Dq-brane system extended in the $x^0, x^1, \ldots, x^q$ directions, and construct Dp-branes along $x^0, \ldots, x^{q-m}$ and $x^{q+1}, \ldots, x^{p+m}$ directions. Generalizing the argument above, we expect that such Dp-branes are classified by $\mathrm{KKO}^{q-1}(\mathbb{R}^{p+m-q}, \mathbb{R}^m)$. Using the isomorphism (4.2), we can again show

$$\mathrm{KKO}^{q-1}(\mathbb{R}^{p+m-q}, \mathbb{R}^m) = \mathrm{KO}(\mathbb{R}^{9-p}) \tag{4.17}$$

which reproduces the correct spectrum of the type-I D-branes as in (4.1).

The explicit tachyon configuration is given by

$$F = u \sum_{\alpha=q-m+1}^{q} \widehat{x}_\alpha \otimes \Gamma^\alpha + u \sum_{\beta=q+1}^{p+m} \partial_\beta \otimes \widehat{\Gamma}^\beta\,, \tag{4.18}$$

where Γ^α and $\widehat{\Gamma}^\beta$ are elements of $M_n(\mathbb{R})\otimes\mathbb{C}^{1,q}_{\rm odd}$ for some n,[11] satisfying

$$\begin{aligned}\Gamma^{\alpha\dagger} &= \Gamma^\alpha\,, & \widehat{\Gamma}^{\beta\dagger} &= -\widehat{\Gamma}^\beta\,, \\ \{\Gamma^\alpha,\Gamma^{\alpha'}\} &= 2\delta^{\alpha\alpha'}\,, & \{\widehat{\Gamma}^\beta,\widehat{\Gamma}^{\beta'}\} &= -2\delta^{\beta\beta'}\,, \qquad \{\Gamma^\alpha,\widehat{\Gamma}^\beta\}=0\,,\end{aligned} \tag{4.19}$$

which are the same as the relations for the generators of $\mathbb{C}^{p+m-q,m}$.

A realization of these Γ^α and $\widehat{\Gamma}^\beta$ representing a single Dp-brane is given as follows. Let us first consider $m=q$ case. One can find the realization for this case generalizing the consideration around (4.8) and (4.9);

$$\widehat{\Gamma}^{q+i} = \gamma_p^i\otimes e_1\,, \qquad (i=1,\dots,p)\,, \tag{4.20}$$

$$\Gamma^i = 1\otimes e_{i+1}\,, \qquad (i=1,\dots,q)\,, \tag{4.21}$$

where e_i $(i=1,\dots,q+1)$ are the generators of $\mathbb{C}^{1,q}$. They are realized as elements of $M_{n_p}(\mathbb{R})\otimes\mathbb{C}^{1,q}_{\rm odd}$, where n_p is the size of the SO(p) gamma matrices γ_p^i listed in table 4.

Then, let us consider general cases with $m\le q$. Note that using the isomorphism $\mathbb{C}^{p,q}=M_{2^{q-m}}(\mathbb{R})\otimes\mathbb{C}^{p+m-q,m}$ (See (A.4).), we can embed $\mathbb{C}^{p+m-q,m}_{\rm odd}$ as $\mathrm{diag}(1,0,\dots,0)\otimes\mathbb{C}^{p+m-q,m}_{\rm odd}$ in $\mathbb{C}^{p,q}_{\rm odd}$. Thus we can realize $\mathbb{C}^{p+m-q,m}_{\rm odd}$ in $M_n(\mathbb{R})\otimes\mathbb{C}^{1,q}_{\rm odd}$ using the realization of $\mathbb{C}^{p,q}$ given by (4.20) and (4.21).

The corresponding configurations in type-IIB theory are given by replacing the real gamma matrices γ_p^i with complex gamma matrices. In any cases, difference of the size of the tachyon between type I and type IIB is again determined by comparing table 4 and table 3, which implies the correct tension for the Dp-brane.

As a check, let us demonstrate the case with $q=9$ and $m=9-p$. In this case, the tachyon is given by

$$F = u\sum_{\alpha=p+1}^{9}\widehat{x}_\alpha\otimes\Gamma^\alpha\,, \tag{4.22}$$

where Γ^α $(\alpha=p+1,\dots,9)$ are elements of $M_n(\mathbb{R})\otimes\mathbb{C}^{1,9}_{\rm odd}$ satisfying the same relations as that for the generators of $\mathbb{C}^{0,9-p}$. As explained above, $\mathbb{C}^{0,9-p}$ can be embedded in $\mathbb{C}^{p,9}=M_{2^p}(\mathbb{R})\otimes\mathbb{C}^{0,9-p}$ as a subalgebra of the form $\mathrm{diag}(1,0,\dots,0)\otimes\mathbb{C}^{0,9-p}$, and the generators of $\mathbb{C}^{p,9}$ can be realized in $M_{n_p}(\mathbb{R})\otimes\mathbb{C}^{1,9}_{\rm odd}$ by (4.20) and (4.21). Note that the relations (A.4) and (A.7) imply $M_{n_p}(\mathbb{R})\otimes\mathbb{C}^{1,9}=M_{32n_p}(\mathbb{R})$. Combining these together, we obtain a realization of Γ^α in $M_{32n_p/2^p}(\mathbb{R})$.

On the other hand, the tachyon configuration representing Dp-brane in D9-brane–anti D9-brane system is given in [2] as (4.22) with real gamma matrices Γ^α of the form (4.6). The matrices Γ^α together with $\Gamma\equiv\mathrm{diag}(1,-1)$ form a real irreducible representation of SO($10-p$) gamma matrices. Therefore the size of these matrices is given by n_{10-p}, where n_p is the size of the real SO(p) gamma matrices listed in table 4. We can check that n_{10-p} is equal to $32n_p/2^p$, in agreement with the above consideration. We can also calculate the ratio of the tension of a type-I Dp-brane to that of a type-IIB Dp-brane, comparing the size of the real and complex SO($10-p$) gamma matrices, which again gives the correct values.

[11] Here we use $\mathbb{C}^{1,q}$ for the definition of the KK-theory group as $\mathrm{KKO}^{q-1}(X,Y)=\mathrm{KKO}(C_0(X,\mathbb{R}),C_0(Y,\mathbb{R})\otimes\mathbb{C}^{1,q})$.

5. Conclusion and discussions

In this paper, we have examined D-branes in type-I string theory. Our main claim is that the D-branes in $X \times Y$ made from the unstable Dp-brane system wrapped on Y are classified by real KK-theory groups $\mathrm{KKO}^{p-1}(X,Y)$. We have explicitly shown that the elements of $\mathrm{KKO}^{p-1}(X,Y)$ can be interpreted in terms of the field theory on the unstable Dp-brane system. It is quite interesting that once we accept the physical interpretation of $\mathrm{KKO}^{p-1}(X,Y)$, we can easily find the field content of type I unstable Dp-brane systems listed in table 2, which was derived by careful consideration of the Ω-projection in [14], by just looking at the Clifford algebra.

The arguments in this paper are also applicable to the USp(32) string theory [24]. The D9-brane–anti D9-brane system of the theory has gauge group of $\mathrm{Sp}\times\mathrm{Sp}$ type, and the relevant K-theory group is $\mathrm{KSp}(X)$ which is isomorphic to $\mathrm{KO}^4(X)$. Therefore, the spectrum of the D-branes are obtained by shifting p to $p+4$ in table 1, which implies that there are stable D$1,5,9$-branes with $\mathbb{Z}$ charges and D$3,4$-branes with $\mathbb{Z}_2$ charges. Accordingly, the KK-theory group which corresponds to the unstable Dp-brane system in this theory is $\mathrm{KKO}^{p+3}(X,Y)$. The field contents of the unstable Dp-brane system should also be obtained by the shift $p \to p+4$ in table 2.

It is not fully clear to us why such Clifford algebra structure appears in the type-I D-branes. As discussed around (2.8), we can use the Clifford algebra $\mathbb{C}^{1,p}$ to define $\mathrm{KKO}^{p-1}(X,Y)$ as $\mathrm{KKO}^{p-1}(X,Y) = \mathrm{KKO}(C_0(X,\mathbb{R}), C_0(Y,\mathbb{R})\otimes\mathbb{C}^{1,p})$. One can imagine that the algebra $\mathbb{C}^{1,p}$ has to do with the fermions on the Dp-brane which transform as a spinor representation of $\mathrm{SO}(1,p)$. Actually, the gauge groups of the systems are derived in [14] so that we can consistently perform the Ω-projection for the fermions created by the open strings connecting the Dp-brane and one of the background D9-branes. It would be interesting to study the relationship between the Clifford algebras and orientifolds. See also [12] for the related discussions.

Another interesting application of the Clifford algebra in type-I D-branes is that we can write down the Chern-Simons terms using the superconnection which are defined by the Clifford algebra. Generalizing the definition of the (complex) superconnection in [25], we can define essentially eight types of real superconnections associated to the corresponding Clifford algebra. Namely, the real superconnection associated to the Clifford algebra $\mathbb{C}^{p,q}$ is given as the formal sum of the tachyon, which is an $M_n(\mathbb{R})\otimes\mathbb{C}^{p,q}_{\mathrm{odd}}$ valued field, and the gauge field, which is an $M_n(\mathbb{R})\otimes\mathbb{C}^{p,q}_{\mathrm{even}}$ valued one-form. More explicitly, we can write it as

$$\mathcal{A} = \sum_{v_n\in\mathbb{C}^{p,q}_{\mathrm{odd}}} T_n v_n + \sum_{w_n\in\mathbb{C}^{p,q}_{\mathrm{even}}} A_n w_n\,, \tag{5.1}$$

where T_n are the $M_n(\mathbb{R})$ valued fields and A_n are the $M_n(\mathbb{R})$ valued one-form fields, and $\{w_n\}$ and $\{v_n\}$ are the basis of $\mathbb{C}^{p,q}_{\mathrm{even}}$ and $\mathbb{C}^{p,q}_{\mathrm{odd}}$, respectively. The first term in the right hand side is nothing but the tachyon F in (3.2), though we have described in a finite rank matrices here. The first term is required to be self-adjoint, as discussed in section 3, while the second term should be anti self-adjoint, since the gauge fields are associated to the Lie algebra of the gauge group. The Chern-Simons term is of the form

$$S_{CS} = \int C \wedge \mathrm{Str}\, e^{\mathcal{F}} \,, \tag{5.2}$$

where C is the sum of RR-fields, $\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$ is the field strength and 'Str' denote the trace of the coefficient of $e_1 e_2 \cdots e_{p+q}$.

The most simple and interesting choice of the unstable Dp-brane systems is the lowest dimensional case $p = -1$, i.e. the matrix theory based on non-BPS D-instantons, which we called K-matrix theory. As a check, we have shown that the spectrum of the flat D-branes constructed in the K-matrix theory are exactly what we expect in the K-theory result [2]. This suggests that we can construct any configurations of D-branes in type-I theory from an infinite number of non-BPS D-instantons. Therefore, it is natural to suppose that we can study dynamics involving various types of D-branes within a single framework of the K-matrix theory, which was one of the main motivation for our previous paper [11]. It would be interesting to explore further in this direction.

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A. Clifford algebra

The Clifford algebra $\mathbb{C}^{p,q}$ is defined as an algebra over $\mathbb{R}$ generated by e_i $(i = 1, \ldots, p+q)$ satisfying

$$e_i e_j + e_j e_i = 0 \qquad (i \neq j) \tag{A.1}$$

$$e_i^2 = -1\,, \quad e_i^* = -e_i \qquad (i = 1, \ldots, p) \tag{A.2}$$

$$e_i^2 = 1\,, \quad e_i^* = e_i \qquad (i = p+1, \ldots, p+q)\,. \tag{A.3}$$

It is a $\mathbb{Z}_2$-graded algebra, defined by the involution $e_i \to -e_i$. And, $\mathbb{C}^{p,q}_{\mathrm{even}}$ and $\mathbb{C}^{p,q}_{\mathrm{odd}}$ denote the sets of even and odd elements with respect to this gradation, respectively.

Here we list some useful isomorphisms among the Clifford algebras.

$$\mathbb{C}^{r+n,s+n} \simeq M_{2^n}(\mathbb{C}^{r,s}) \tag{A.4}$$

$$\mathbb{C}^{r,s} \simeq \begin{cases} M_{2^s}(\mathbb{C}^{r-s,0}) & (r > s) \\ M_{2^r}(\mathbb{C}^{0,s-r}) & (r < s) \end{cases} \tag{A.5}$$

$$\mathbb{C}^{r+4,s} \simeq \mathbb{C}^{r,s+4}\,, \qquad \mathbb{C}^{r,s+1} \simeq \mathbb{C}^{s,r+1}\,, \tag{A.6}$$

$$\mathbb{C}^{r+8,s} \simeq \mathbb{C}^{r,s+8} \simeq M_{16}(\mathbb{C}^{r,s})\,, \tag{A.7}$$

$$\mathbb{C}^{n+1,0}_{\mathrm{even}} \simeq \mathbb{C}^{0,n+1}_{\mathrm{even}} \simeq \mathbb{C}^{n,0}\,. \tag{A.8}$$

n	$\mathbb{C}^{n,0}$	$\mathbb{C}^{0,n}$	$\mathbb{C}^{n,0}_{\text{even}} \simeq \mathbb{C}^{0,n}_{\text{even}}$	Gauge group
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	O
2	$\mathbf{H}$	$M_2(\mathbb{R})$	$\mathbb{C}$	U
3	$\mathbf{H} \oplus \mathbf{H}$	$M_2(\mathbb{C})$	$\mathbf{H}$	Sp
4	$M_2(\mathbf{H})$	$M_2(\mathbf{H})$	$\mathbf{H} \oplus \mathbf{H}$	Sp $\times$ Sp
5	$M_4(\mathbb{C})$	$M_2(\mathbf{H}) \oplus M_2(\mathbf{H})$	$M_2(\mathbf{H})$	Sp
6	$M_8(\mathbb{R})$	$M_4(\mathbf{H})$	$M_4(\mathbb{C})$	U
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_8(\mathbb{C})$	$M_8(\mathbb{R})$	O
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	O $\times$ O

Table 6: Clifford algebra

The isomorphism (A.4) is given by

$$\begin{aligned}
\mathbb{C}^{r+1,s+1} &\to M_2(\mathbb{R}) \otimes \mathbb{C}^{r,s} && \text{(A.9)}\\
e_1 &\to \sigma_1 \otimes e_1 && \text{(A.10)}\\
e_2 &\to \sigma_3 \otimes e_1 && \text{(A.11)}\\
e_{i+1} &\to 1_2 \otimes e_i \qquad (i = 2, \ldots, r) && \text{(A.12)}\\
e_{r+2} &\to \epsilon \otimes e_1 && \text{(A.13)}\\
e_{r+2+i} &\to 1_2 \otimes e_{r+i} \qquad (i = 1, \ldots, s), && \text{(A.14)}
\end{aligned}$$

where $\epsilon = i\sigma_2$. The isomorphism (A.7) is given by

$$\begin{aligned}
\mathbb{C}^{r+8,s} &\to M_{16}(\mathbb{R}) \otimes \mathbb{C}^{r,s} && \text{(A.15)}\\
e_i &\to \gamma_9^i \otimes e_1 \qquad (i = 1, \ldots, 9) && \text{(A.16)}\\
e_{i+8} &\to 1_{16} \otimes e_i \qquad (i = 2, \ldots, r+s)\,, && \text{(A.17)}
\end{aligned}$$

where γ_9^i $(i = 1, \ldots, 9)$ are SO(9) gamma matrices represented as real symmetric 16×16 matrices.

From these relations, it is sufficient to know $\mathbb{C}^{n,0}$ and $\mathbb{C}^{0,n}$ $(n = 1, 2, \ldots, 8)$ to obtain the others. They are listed in table 6.

As an example, let us consider $\mathbb{C}^{0,3}$, which is used in section 3.7. The algebra $\mathbb{C}^{0,3}$ is generated by e_1, e_2 and e_3 satisfying (A.1)$\sim$(A.3). They are faithfully represented by Pauli matrices as $e_1 = \sigma_1$, $e_2 = \sigma_2$ and $e_3 = \sigma_3$. Each element $A \in \mathbb{C}^{0,3}$ is of the form

$$\begin{aligned}
A &= a + be_1 + ce_2 + de_3 + ee_1e_2 + fe_1e_3 + ge_2e_3 + he_1e_2e_3\\
&= \begin{pmatrix} a + hi + d + ei & b + gi - (f + ci) \\ b + gi + f + ci & a + hi - (d + ei) \end{pmatrix}, \qquad \text{(A.18)}
\end{aligned}$$

where $a, b, \ldots, f \in \mathbb{R}$. Since (A.18) is a general element of the algebra of complex 2×2 matrices $M_2(\mathbb{C})$, we obtain $\mathbb{C}^{0,3} \simeq M_2(\mathbb{C})$. The even elements are

$$\begin{aligned}
A_{\text{even}} &= a + ee_1e_2 + fe_1e_3 + ge_2e_3\\
&= \begin{pmatrix} a + ei & -f + gi \\ f + gi & a - ei \end{pmatrix} = a + gi\sigma_1 - fi\sigma_2 + ei\sigma_3\,. \qquad \text{(A.19)}
\end{aligned}$$

Since $i\sigma_i$ ($i = 1, 2, 3$) can be represented by the generators of quaternion as $i\sigma_1 \to i$, $i\sigma_2 \to j$ and $i\sigma_3 \to -k$, we obtain $\mathbb{C}^{0,3}_{\text{even}} \simeq \mathbf{H}$.

Note that we can immediately read the gauge groups of the type I unstable D-brane systems from table 6. As explained in section 3, the gauge groups consist of unitary operators which are real operators tensored by even elements of the corresponding Clifford algebra. Recall that a unitary matrix whose matrix elements are $\mathbb{R}$, $\mathbb{C}$ or $\mathbf{H}$-valued is an orthogonal, unitary or symplectic matrix, respectively. (See appendix B for the symplectic case.) Therefore, the gauge group of the system is O, U or Sp, when the even elements of the corresponding Clifford algebra is $M_n(\mathbb{R})$, $M_n(\mathbb{C})$ or $M_n(\mathbf{H})$, respectively, as listed in the last column of table 6.

B. Quaternionic representation of Sp group

Let us recall the definition of the Sp group.[12] The (unitary) Sp group consists of matrices $g \in M_{2N}(\mathbb{C})$ satisfying

$$g^\dagger g = 1\,, \qquad g^T \mathbf{J} g = \mathbf{J} \tag{B.1}$$

where

$$\mathbf{J} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \tag{B.2}$$

(B.1) is equivalent to the condition that g is of the form

$$g = g_a + g_b i\sigma_1 + g_c i\sigma_2 + g_d i\sigma_3 \tag{B.3}$$

$$= \begin{pmatrix} g_a + i g_d & g_c + i g_b \\ -g_c + i g_b & g_a - i g_d \end{pmatrix}, \tag{B.4}$$

satisfying $g^\dagger g = 1$, where g_a, g_b, g_c and g_d are real $N \times N$ matrices.

It is useful to represent $i\sigma_i$, by the generators of quaternion as $i\sigma_1 \to i$, $i\sigma_2 \to j$ and $i\sigma_3 \to -k$. Then (B.3) can be represented as a quaternionic matrix

$$g = g_a + g_b i + g_c j - g_d k \in M_N(\mathbf{H})\,. \tag{B.5}$$

In this way, the Sp group can be represented as unitary quaternionic matrices.

The adjoint representation of the Sp-group is equivalent to the symmetric tensor representation $\square\!\square$. It is given by the anti-hermitean matrices $X \in M_{2N}(\mathbb{C})$ satisfying

$$(\mathbf{J}X)^T = \mathbf{J}X\,. \tag{B.6}$$

The action of a element of the group g is given by $X \to g^\dagger X g$. In the quaternionic representation, this condition is equivalent as saying that $X \in M_N(\mathbf{H})$ is quaternionic anti-hermitean matrix. Though this statement can be shown explicitly using (B.6), it is obvious from the consideration above since the adjoint representation is given by the Lie algebra of the group.

[12] The Sp group we consider in this paper is unitary symplectic group, which is often denoted by USp($2N$).

The anti-symmetric tensor representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ is given by hermitean matrices $X \in M_{2N}(\mathbb{C})$ satisfying

$$(\mathbf{J}X)^T = -\mathbf{J}X\,, \tag{B.7}$$

which can be written as

$$X = \begin{pmatrix} a+di & -b+ci \\ b+ci & a-di \end{pmatrix} = a + bi\sigma_1 + ci\sigma_2 + di\sigma_3\,, \tag{B.8}$$

where $a, b, c, d \in M_N(\mathbb{R})$ satisfy $a = a^T$, $b = -b^T$, $c = -c^T$ and $d = -d^T$. (B.8) is represented as

$$X = a + bi + cj - dk\,, \qquad X^\dagger = X \tag{B.9}$$

in the quaternionic representation.

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